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ON THE STABILITY OF NON-LINEAR STATIONARY WAVES*

A.L. ZHUPIEV and YU.V. MIKHLIN

The stability, to a first approximation, of stationary travelling waves is studied for the Klein-Gordon equation. The corresponding equations in variations are reduced to the form of equations with singularities. The cases when the equations in variations represent the Lamé equations, are singled out. In the latter cases an analysis of the stability to a first approximation can be carried out to conclusion.

1. Consider the non-linear Klein-Gordon equation

$$\varphi_{xx}-\varphi_{tt}=p(\varphi)$$

where $p(\varphi)$ is an analytic function. The stationary waves represent particular solutions of the form $\varphi=\Phi(\xi)$, $\xi=x-ut$ (we assume that the function $\Phi(\xi)$ is bounded at infinity), defined by the equation

$$(1-u^2)\Phi_{\xi\xi}=p(\varphi)\ (u\neq 1) \tag{1.1}$$

with the first analytic integral

$$\tfrac{1}{2}(1-u^2)\Phi_\xi^2=F+P(\Phi);\quad F=\text{const},\quad P(\Phi)=\int\limits_0^\Phi p(z)\,dz \tag{1.2}$$

The equation in variations for the stationary waves (in the ξ, t variables) has the following form:

$$y_{\xi\xi}(1-u^2)+2y_{\xi t}u-y_{tt}=p_\Phi(\Phi)\,y$$

Applying a Laplace transform in t, we obtain an equation for the transformations $V(\xi,s)$ where s is the transformation parameter. Carrying out the substitution $V=e^{A\xi}W$ where $A=-su/(1-u^2)$, we shall write the following equation for W with coefficients variable in ξ :

$$2W_{\xi\xi}(1-u^2)+W(B-p_\Phi(\Phi))=0,\ B=s^2/(u^2-1) \tag{1.3}$$

Let us consider the linear stability in t of the stationary wave in the class of perturbations bounded in ξ /1/. In this case the existence of such perturbations at $s^2>0$ indicates that the wave is unstable.

Henceforth, it will be more convenient to use new variables connected with the solution in question. If we replace ξ as the independent variable by Φ, then, using (1.1) and (1.2), we obtain in place of (1.3),

$$2W_{\Phi\Phi}(F+P(\Phi))+W_\Phi p(\Phi)+W(B-p_\Phi(\Phi))=0 \tag{1.4}$$

Choosing $z=\Phi^2$, as the independent variable we obtain

$$8W_{zz}z(F+P(\sqrt{z}))+W_z[4(F+P(\sqrt{z}))+2\sqrt{z}p(\sqrt{z})]+W(B-p_\Phi(\sqrt{z}))=0 \tag{1.5}$$

which is appropriate only for the even functions $p(\Phi)$).

Finally, if we use the kinetic energy of the stationary wave $K=F+P(\Phi)$, as the independent variable, the equation in variations will become

$$2W_{kk}Kp^2(\Phi(K))+W_k[2Kp_\Phi(\Phi(K))+p^2(\Phi(K))]+W[B-p_\Phi(\Phi(K))]=0 \tag{1.6}$$

Let us separate the classes of functions $P(\Phi)$ for which the equations obtained represent the Lamé equations. In this case we can determine the domains of boundedness and unboundedness of the solutions of the variational equations

a) $$P=D_1\sin\Phi-D_2\cos\Phi,\ p=D_1\cos\Phi+D_2\sin\Phi$$

(the Gordon sine equation). Here we must use equation (1.6), which is reduced to the Lamé equation in its standard form /2/

$$W_{kk}+\frac{1}{2}W_k\left(\frac{1}{K-\gamma_1}+\frac{1}{K-\gamma_2}+\frac{1}{K-\gamma_3}\right)+W\frac{H-n(n+1)K}{4(K-\gamma_1)(K-\gamma_2)(K-\gamma_3)}=0 \tag{1.7}$$

$$\gamma_1=0,\ \gamma_{2,3}=F\pm 1,\ H=2(F-B),\ n=1$$

The last equation shows that the (F,B) parameter plane contains one finite and one infinite domain of unboundedness of the solutions /2/

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b) $$P = D_1 \operatorname{sh} \Phi + D_2 \operatorname{ch} \Phi$$

Using again the equations (1.6) we arrive at (1.7), where $\gamma_1 = 0, \gamma_{2,3} = F \pm 1$, $H = 2(F + B)$, $n = 1$.

c) $$P = \alpha_2\Phi^2/2 + \alpha_4\Phi^4/4$$

Here it is best to use the equation in variations in the form (1.5). In this case it will also be reduced to the Lamé equation of the form (1.7) where

$$z \equiv K,\ \gamma_1 = 0,\ \gamma_{2,3} = -\frac{\alpha_2}{\alpha_4} \pm \sqrt{\left(\frac{\alpha_2}{\alpha_4}\right)^2 - 4\frac{F}{\alpha_4}},\quad H = 2\frac{B - \alpha_2}{\alpha_4},\quad n = 2$$

The last equation indicates that two finite and one infite domain in which the solutions are unbounded, exist in the parameter space

d) $$P = (\Phi - \gamma_1)(\Phi - \gamma_2)(\Phi - \gamma_3) - F$$

In the case we must use equation (1.4), which reduces to the standard form (1.7) when

$$\Phi \equiv K,\ H = B + 2(\gamma_1 + \gamma_2 + \gamma_3),\ n = 2$$

We note that we have shown here all cases in which the variational equation of the stationary waves of the Klein-Gordon equation is reduced to the form of the Lamé equations.

2. The regions in which the solutions of the Lamé equations are bounded and unbounded, are known for $n = 1, 2$ /3/, and this makes it possible to study the stability of the stationary waves with respect to time. The boundaries of the corresponding regions for $n = 1$ are shown in Fig.1, and for $n = 2$ in Fig.2. Here $\lambda = (e_2 - e_3)/(e_1 - e_3)$, $\mu = (H - n(n+1)e_3)/(e_1 - e_3)$ $(e_1, e_2, e_3$ correspond to $\gamma_1, \gamma_2, \gamma_3)$ introduced earlier, which are arranged in such an order that the inequalities $e_1 > e_2 > e_3)$ hold. The regions of boundedness are hatched, and curves 1–5 in Fig.2 are described by the equations

1) $\mu = \mu^+(\lambda)$, 2) $\mu = 4 + \lambda$, 3) $\mu = 1 + 4\lambda$, 4) $\mu = 1 + \lambda$, 5) $\mu = \mu^-(\lambda)$
$(\mu^\pm(\lambda) = 2(1 + \lambda \pm \sqrt{\lambda^2 - \lambda + 1}))$

The stability in time is analysed as follows. Taking every solution in the form of a stationary wave, we find out whether bounded solutions of the equation in variations exist for $s^2 > 0$, which would indicate the instability. If no such solutions exist for $s^2 > 0$, we have stability to a first approximation.

As an example we shall carry out the corresponding analysis for the Gordon sine equation for case a) where we write $D_1 = 0, D_2 = 1$ for simplicity. The regions of boundedness and unboundedness shown in Fig.1 are also shown in Fig.3 in terms of the parameters B, F; $B^\pm = {}^1/_2(F \pm 1)$.

Let us single out the following classes of solutions of the initial equation /4/:

periodic waves for $|F| < 1, u^2 - 1 > 0$ or $u^2 - 1 < 0$;

spiral waves for $F < -1, u^2 - 1 > 0$;

spiral waves for $F > 1, u^2 - 1 < 0$.

It is clear (Fig.3) that when $F > -1$, the bounded solutions exist for $B > 0$ as well as for $B < 0$ (here we have $s^2 \leqslant 0$ and $s^2 > 0$), and this implies that the periodic ($|F| < 1$) and spiral $(F > 1, u^2 - 1 < 0)$ waves are unstable in the class of the perturbations bounded at infinity.

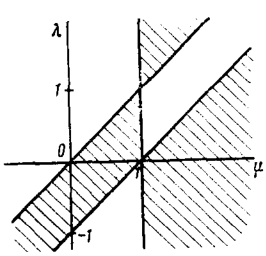

Fig. 1

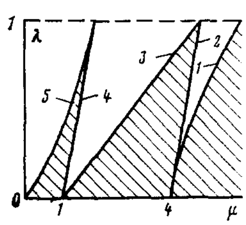

Fig.2

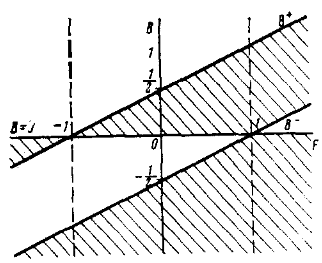

Fig.3

For the spiral waves $(F < -1, u^2 - 1 > 0)$ the bounded solutions exist only when $B < 0$ $(s^2 \leqslant 0$, since $u^2 - 1 > 0)$, which implies the stability to a first approximation.

We shall consider the case b) as another example. The conditions of boundedness of the solutions of the equations in variations (Fig.2) are written in the form

$$\mu^-(\lambda) < \mu < 1 + \lambda,\ 1 + 4\lambda < \mu < 4 + \lambda,\ \mu^+(\lambda) < \mu \tag{2.1}$$

Let $F = 0,\ \alpha_2 = \alpha_4 = 1$. Then

$$e_1 = e_2 = 0,\ e_3 = -2,\ \lambda = 1,\ \mu = B + 5$$

The conditions boundedness of (2.1) lead to the inequality $B > 1$ or $s^2 > u^2 - 1$ when $u^2 - 1 > 0$, and $s^2 < u^2 - 1$ when $u^2 - 1 < 0$. It is clear that the condition $s^2 > 0$ (instability) may hold when $u^2 - 1 > 0$, and the solution is stable when $u^2 - 1 < 0$.

Let $F = -3/16,\ \alpha_2 = -\alpha_4 = 1$. Then

$$e_1 = 3/2,\ e_2 = 1/2,\ e_3 = 0,\ \lambda = 1/3,\ \mu = -4/3B + 1/3$$

The conditions of boundedness of (2.1) lead to the system of inequalities $0.9 < -4/3B < 1$; $2 < -4/3B < 4$; $4.1 < -4/3B$, and this implies that $B < 0$. It is clear now that $s^2 > 0$ when $u^2 - 1 < 0$ (instability) and $s^2 < 0$ when $u^2 - 1 > 0$ (stability).

Finally, let $\alpha_2 = -\alpha_4 = 4,\ F = 3$. Then $e_1 = 3,\ e_2 = 0,\ e_3 = -1,\ \lambda = 1/4,\ \mu = -B/8 + 2$.

Conditions (2.1) yield the system of inequalities

$$6 < B < 10.4;\ -20 < B < 0;\ -18.4 < B$$

Since B may take positive, as wells as negative values, it is clear that we can have $s^2 > 0$ also when $u^2 - 1 < 0$ as well as when $u^2 - 1 > 0$, and this implies the instability.

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